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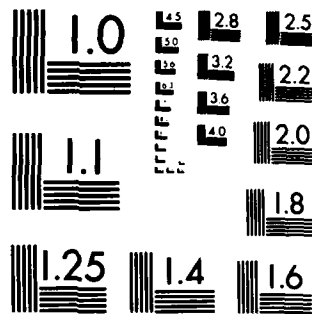
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FINAL TECHNICAL REPORT

Material Instabilities in Solids  
AFOSR-86-0184

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Abstract. The principal investigator considered a number of mathematical problems that may help to explain material failures in polymers and ductile metals. In particular, useful results were obtained concerning the initiation and dynamic growth of holes, the static instability and surface cracking of a single hole and the linearized stability of a free surface in an elastic material.

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## SUMMARY

Experiments on elastomers and ductile metals by other researchers have indicated that a major failure mechanism in such materials is that of void formation and coalescence (as loads are applied small holes appear, grow larger, and combine to form cracks). In aluminum, in particular, an alternative failure mechanism is due to surface wrinkling.

The principal investigator considered a number of mathematical problems that could help to explain the aforementioned material failures. In particular, useful results were obtained concerning: the initiation and dynamic growth of voids; the static instability and surface cracking of a single void; and the linearized stability of a free surface in an elastic material.

**Dynamic Formation of Voids.** A new and surprising result was obtained for the equations of nonlinear elastodynamics. It was shown that a steady state equilibrium solution is not the only solution that starts at the steady state with no initial velocity. The constructed solutions are radial similarity solutions that open up new holes in the material and dissipate energy at their interface with the equilibrium solution. These solutions were used to show that in order for certain materials to dissipate the maximal amount of energy the material will attempt to open new voids at all points of sufficiently large strain. This result is in qualitative agreement with the experimental observations in such materials.

**Static Instability of Voids.** It was shown that a single spherical void is not the energy minimizer for a class of nonlinearly elastic materials. In particular, the formation of secondary cracks on the void surface should yield solutions of lower energy. The linking of these secondary cracks between two distinct voids may be the mechanism for void coalescence that causes final material failure. In obtaining this result a new

test for the static stability of solutions for a homogeneous material was discovered. Roughly speaking this result says that in order for a solution to be stable with respect to new void formation each of the homogeneous deformations obtained by applying the deformation gradient at one point to the whole material must be stable with respect to new void formation. It is expected that this result will be useful in determining the stability of solutions that other researchers have constructed for fracture problems.

Surface Instabilities. Necessary and sufficient conditions for the linearization stability of an elastic surface were obtained. In particular it was found that the Legendre-Hadamard condition, Agmon's condition and a new restricted rank-two-convexity condition are important. It is expected that this result can be used to analyze constitutive relations to determine which are appropriate for explaining surface wrinkling.

#### Publications

1. Nonuniqueness for a Hyperbolic System: Cavitation in Nonlinear Elastodynamics (with K. A. Pericak-Spector). To appear: Archive for Rational Mechanics and Analysis. Accepted.
2. Necessary Conditions at the Boundary for Minimizers in Finite Elasticity (with H. C. Simpson). To be submitted to the Archive for Rational Mechanics and Analysis.
3. On the Formation of Filamentary Voids in Elastic Materials (with R. D. James). In preparation.
4. Some Necessary Conditions at a Phase Boundary for Minimizers in Finite Elasticity (with H. C. Simpson). In preparation. To be submitted to the Journal of Elasticity.

Interactions.

1. "On Nonuniqueness for a Hyperbolic System: Cavitation in Nonlinear Elastodynamics" and "On Radial and Non-radial Minimizers: Cavitation in Nonlinear Elastostatics." Presented to the Center for Applied Mathematics, Purdue University, West Lafayette, IN during March 1987. Among those in attendance were: P. Bauman, A. Friedman, J. Necas, N. Owen, D. Phillips, and T. Tzavaras.
2. "On Void Formation in Elastic Materials." Presented to the Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN during May 1987. Among those in attendance were: C. Calderer, M. Chipot, J. Ericksen, R. Fosdick, R. James, D. Kinderlehrer, M. Luskin, P. Sethna, and W. Warner.
3. "On Fracture via Elastodynamic Cavitation." Presented to the Sixth International Conference on Mathematical Modelling, St. Louis, MO during August 1987. Among those in attendance were M. Berger, J. Maddocks, A. Nachman, and A. Novick-Cohen.
4. "On Void Formation in Elastic Materials." Presented to the Mathematical Sciences Institute, Cornell University, Ithaca, NY during October 1987. Among those in attendance were: Y. Chen, T. Healey, P. Holmes, C. Hui, J. Jenkins, E. MacMillan, L. Payne, and H. Weinberger.
5. "On Void Formation in Elastic Materials." Presented to the Center for Mathematical Science, University of Wisconsin, Madison, WI during November 1987. Among those in attendance were W. Drugan, M. Gurtin, J. Nohel, R. Meyer, R. Rogers, and M. Slemrod.

Detailed Description of Results Obtained.

Nonuniqueness for a Hyperbolic System: Cavitation  
 in Nonlinear Elastodynamics

by

K. A. Pericak-Spector & Scott J. Spector

We let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , and consider the problem: Find  
 $u: [0, T] \times \Omega \rightarrow \mathbb{R}^n$  that satisfies

$$\begin{aligned} \operatorname{div} S(\nabla u) &= u_{tt} && \text{in } \Omega \times [0, T], \\ u(x, t) &= \lambda x && \text{for } x \in \partial\Omega \text{ and } t \in [0, T], \\ u(x, 0) &= \lambda x && \text{for } x \in \Omega, \\ u_t(x, 0) &= 0 && \text{for } x \in \Omega, \end{aligned}$$

where

$$S(F) := F + h'(\det F) \operatorname{adj} F$$

with  $h'' > 0$ ,  $h''' < 0$ ,  $h'(u) \rightarrow -\infty$  as  $u \rightarrow 0^+$ , and  $h'(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$ .

We prove that there are  $\lambda_1 \rightarrow +\infty$  such that the above problem, with  $\lambda = \lambda_1$ , has multiple solutions. Moreover we show that each such solution is admissible according to the standard entropy criterion.

The solutions that we construct are superpositions of special cavitation similarity solutions; that is, solutions of the form

$$u(x, t) := \frac{\varphi(s)}{s}(x - x_0), \quad s := |x - x_0|/(t - t_0),$$

for  $t \geq t_0$ , where  $\varphi$  is continuous, piecewise  $C^2$  and satisfies

$$\varphi(0) > 0, \quad \varphi(s) = \lambda s \text{ for } s \geq \sigma,$$

for some  $\sigma > 0$ . Thus each such solution is discontinuous, opening a hole in the region  $\Omega$ .



An important unresolved problem is the precise smoothness of  $\varphi$  (numerical experiments indicate that  $\varphi$  is not  $C^1$ ). If  $\varphi$  is not  $C^1$  then the solution  $u$  will contain a shock that dissipates energy. In this case our results admit an interesting physical interpretation. Consider the tip of a crack in an elastic material that is undergoing loading. If the load is sufficiently large then, according to our results, we expect the material to fail; that is, in order to dissipate as much energy as possible the material must attempt to open a cavity at every point near the crack tip. Thus, in practice, we expect to see a large number of cavities form. Of course our results are not directly applicable to this situation since the stress field is not precisely uniform triaxial tension near the tip of a crack. However, our explanation of fracture initiation is in qualitative agreement with experimental observations in certain ductile metals.

The above interpretation assumes that cavities appear spontaneously. Another explanation for their appearance is that they grow from submicroscopic lattice defects. In elastostatics this distinction is unimportant since Sivaloganathan has shown that the load required to produce a new cavity is of the same order as the load range where a small cavity will grow rapidly. In elastodynamics no such result is yet available.

Finally, we note that our work was motivated by the fundamental paper of Ball on cavitation in elastostatics. Ball showed that for a large class of stored energy functions the equilibrium solution  $u(x,t) = \lambda x$  is not the global minimizer of the energy for large  $\lambda$ . He also showed that among radial deformations the global minimizer exists and exhibits cavitation. Thus our results are a dynamic version of Ball's for the special constitutive relation given above.

Necessary Conditions at the Boundary  
 for Minimizers in Finite Elasticity

by

Henry C. Simpson

&

Scott J. Spector

In this paper we derive pointwise algebraic conditions on the elasticity tensor  $C(x, \nabla f(x))$  that are necessary for a deformation  $f$  to be a local minimizer of the energy of an elastic body that is subjected to dead loads. In particular we show that the Legendre-Hadamard condition, Agmon's condition, and the new condition: if, for some vector  $e$ ,

$$e \otimes n \cdot C[e \otimes n] = 0$$

then

$$C[e \otimes n] = 0$$

are necessary conditions. Here  $n = n(x)$  is the outward unit normal to the boundary at any point  $x$  where the deformation is not prescribed and  $C = C(x, \nabla f(x))$  where  $C = \partial^2 W / \partial (\nabla f)^2$ ; the second derivative of the stored energy  $W$ .

We also show that the three aforementioned conditions are sufficient for the nonnegativity of the second variation of the energy of a homogeneous deformation of a homogeneous body that, in the reference configuration, has the shape of a half-ball (The crucial geometric assumption is that the boundary of the body contains a portion of a hyperplane that is the only surface upon which the deformation is not prescribed.) with the deformation prescribed on the curved surface of the body.

A long standing problem in the calculus of variations is that of finding pointwise algebraic conditions on  $C$  that are necessary and sufficient for the nonnegativity of the second variation, i.e.,

$$\delta^2 E_f(u) := \int_{\Omega} \nabla u(x) \cdot C(x, \nabla f(x)) [\nabla u(x)] dx \geq 0 \quad (1)$$

for all variations  $u$ . It is well-known that a necessary condition for (1) to be satisfied is the Legendre-Hadamard condition, that is,

$$a \otimes b \cdot C[a \otimes b] \geq 0 \quad (2)$$

for all points  $x \in \bar{\Omega}$  and all vectors  $a$  and  $b$ . A well-known condition that is sufficient for (1) is that  $C$  be positive semi-definite, that is,

$$M \cdot C[M] \geq 0 \quad (3)$$

for all points  $x \in \bar{\Omega}$  and matrices  $M$ .

In the special case when  $C(x, \nabla f(x)) \equiv C$  is independent of  $x$  more precise results are known. In particular van Hove proved that (2) is necessary and sufficient for the nonnegativity of the second variation provided that all variations satisfy  $u = 0$  on the boundary of  $\Omega$ . If all smooth functions are admitted as variations then the choice  $u(x) = Mx$  shows that (3) is necessary and sufficient for (1). Thus the boundary conditions imposed upon the variations are a crucial part of this problem.

We note that our results are related to the existing work on surface instabilities in a half-space. In such problems one is interested in finding algebraic conditions on the elasticity tensor such that the complementing (Lopatinsky-Shapiro) condition fails. Thus the linearized equilibrium equations in a half-space  $\mathcal{H}$  admit a nontrivial bounded exponential solution, i.e.,

$$\begin{aligned} \operatorname{div} C[\nabla w] &= \alpha^2 w & \text{in } \mathcal{H}, \\ C[\nabla w]n &= 0 & \text{on } \partial\mathcal{H}, \end{aligned} \quad (4)$$

where  $\alpha = 0$ ,  $n$  is the outward unit normal to  $\partial\mathcal{H}$ , and

$$w(x) = z(-x \cdot n) \exp(i x \cdot t)$$

with  $z$  bounded and  $t$  tangent to  $\partial\mathcal{H}$ .

One of the necessary conditions that we derive, Agmon's condition, is the requirement that (4) not admit a nontrivial

bounded exponential solution for any  $\alpha \neq 0$ . The failure of this condition induces a more severe instability than the failure of the complementing condition since it is possible for the complementing condition to fail at the unique global minimizer of the energy while, as a consequence of our results, the failure of Agmon's condition implies that the underlying deformation is not even a local minimizer.

Finally, we note that our results are also related to those of Chen who considered the problem of finding pointwise conditions on the (constant) elasticity tensor  $C$  that are necessary and sufficient for the nonnegativity of the second variation when the body is a thin plate and the variations satisfy  $u = 0$  on the edges of the plate. In particular Chen proved that a restricted rank-two convexity condition, i.e.,

$$[a \otimes b + e \otimes n] \cdot C[a \otimes b + e \otimes n] \geq 0 \quad (5)$$

for all vectors  $a$ ,  $b$ , and  $e$ , is a sufficient condition for (1) to be satisfied for such a plate and that (5) is also necessary for (1) to be satisfied for arbitrarily thin plates. However, (5) is not necessary if the size of the plate remains fixed.

The new necessary conditions that we derive can also be viewed as a restricted rank-two convexity condition since it together with the Legendre-Hadamard condition, (2), imply that if for some vector  $e$ ,

$$e \otimes n \cdot C[e \otimes n] = 0$$

then (5) is satisfied for all vectors  $a$  and  $b$ .

On the Formation of Filamentary Voids in Elastic Materials

by

R. D. James & S. J. Spector

In 1958 Gent & Lindley observed an unusual rupture process in short rubber cylinders that were bonded at their ends to parallel steel plates and pulled in tension. At a load that was, in many cases, less than a fourth of the ultimate breaking load, they observed the appearance of small, approximately spherical holes toward the ends of the test piece.

An analysis of this phenomena was made in 1982 in a fundamental paper by Ball who considered the problem of minimizing the stored energy of a homogeneous, isotropic ball subject to homogeneous boundary conditions. He found that, for appropriate materials and sufficiently large loads, the global minimizer of the energy among spherically symmetric deformations contained a spherical cavity.

To our knowledge all analyses of void formation in elastic materials have involved problems with spherical symmetry in which all deformations that compete for minimum energy have the special form:

$$f(x) = \frac{r(R)}{R} x, \quad R = |x|.$$

To actually predict the formation of voids at a stress concentration two questions arise:

1. Are the spherically symmetric solutions with holes minimizers of the energy when arbitrary (non-spherically symmetric) deformations are allowed to compete for a minimum?
2. What information from the analyses of homogeneous boundary value problems can be carried over to inhomogeneous deformations?

In this paper we give partial answers to these questions. Regarding Question 2 we show that if a void of any kind (not

necessarily spherically symmetric) reduces the energy in a homogeneous boundary value problem boundary then one can find a closely related deformation that reduces the energy in an inhomogeneous problem.

The answer to Question 1 clearly depends on the energy function. We show that for a large class of realistic energy functions the answer is no. Our method is based on the following ideas. Consider first the spherical problem with boundary condition  $f(x) = \lambda x$  for  $|x| = 1$ . The homogeneous deformation satisfying these boundary conditions,  $f(x) = \lambda x$  for  $|x| \leq 1$ , involves large volume changes when  $\lambda$  is large. A material may find it energetically unfavorable to undergo such a large volume change and will instead open a spherical hole at its center. The principal stretches in such a deformation typically are of the form:

$$\alpha(R), \quad \lambda(R), \quad \lambda(R)$$

with  $\alpha\lambda^2$  bounded as  $R \rightarrow 0^+$ ,  $\lambda(R) \rightarrow +\infty$  as  $R \rightarrow 0^+$ , and  $\alpha(R) \rightarrow 0$  as  $R \rightarrow 0^+$ . Our analysis involves the same idea carried one step further. A deformation with the aforementioned principal stretches might also be judged energetically unfavorable because of the large change in area  $\lambda^2(R)$  as  $R \rightarrow 0^+$ . In particular we show that this is indeed true for some realistic energy functions and that these materials would prefer to have principal stretches

$$\alpha(R), \quad \lambda_1(R), \quad \lambda_2(R)$$

with  $\lambda_1\lambda_2 < \lambda^2$ . In order to introduce a competitor with these principal stretches we construct an extremely thin, short filamentary void just outside the boundary of the hole. This filamentary void then reduces the total energy of the material.

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